School Timetabling for Quality Student and Teacher Schedules

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Abstract:

The school timetabling problem, although less complicated than its counterpart for the university, still provides a ground for interesting and innovative approaches that promise solutions of high quality. In this work, a Shift Assignment Problem is solved first and work shifts are assigned to teachers. In the sequel, the actual Timetabling Problem is solved while the optimal shift assignments that resulted from the previous problem help in defining the values for the cost coefficients in the objective function. Both problems are modelled using Integer Programming and by this combined approach we succeed in modelling all operational and practical rules that the Hellenic secondary educational system imposes. The resulting timetables are conflict free, complete, fully compact and well balanced for the students. They also handle simultaneous, collaborative and parallel teaching as well as blocks of consecutive lectures for certain courses. In addition, they are highly compact for the teachers, satisfy the teachers´ preferences at a high degree, and assign core courses towards the beginning of each day.

Keywords: *Educational Timetabling; Integer Programming; Shift assignment*

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1.0 Introduction

School Timetabling as a term refers to the construction of weekly timetables for schools of secondary education (Schaerf, 1999). Formally, in this paper it is defined as the assignment of triplets consisting of teachers, classes and courses to certain time-periods of each day of the week, subject to constraints and in such way that a cost function is minimized. For the Hellenic gymnasium (the lower levels of secondary education, i.e. high school grades 7 to 9), while the timetabling process was carried out in a simpler way a few years back (Birbas et al., 1997a; 1997b; 1999), current educational trends require more complicated schedules to cover special needs and to provide certain flexibility with the curricula offered to the students. Quality, on the other hand, which is an ill-defined feature, is always required from the actual timetables. While it is not easy to embody quality into the modelling process, it is even more difficult to quantify it. Many of the quality issues, set forth by the institutions, are turned into hard requirements and the timetablers have to abide by them. The remaining issues are usually declared as "desired" and then it is important to tune the solution method so that it approaches as close as possible to solutions of "high quality".

According to the Hellenic educational system, the students in the gymnasium form class-sections and all students of a given class-section follow the same schedule for most part of the week. However, for some time periods certain class-sections split into smaller ones to attend different courses or merge with others to form new class-sections just for one or two time periods. These planning techniques are necessary to handle elective courses or courses like foreign languages taught at different levels or courses that require specially equipped rooms that cannot handle the full number of students in a class-section. Although necessary, such scheduling requirements create extra difficulties in the timetabling process resulting in timetables that are often quite unsatisfactory.

In order to handle the flexibility introduced in the school curricula, our efforts towards automatic timetabling had to evolve also. Compared to our previous approach that suggested a single Integer Programming (IP) model (Birbas et al., 1997a; 1997b), the new modelling approach involves two stages. The *first stage* is preliminary and assigns work shifts to teachers. In the *second stage*, the actual timetabling problem is solved, while guided but not constrained by the shift assignments. The central idea of the new approach is to create an initial temporal assignment for the teachers based on which part of the day they prefer to teach; then the timetabling problem is used to specify the exact allocation of courses, teachers and class-sections to time periods and days. It is important to note that the timetabling problem is solved as a whole and does not break into sub-problems, thus the solution will be optimal. Of course the notion of "work shift" does not exist inherently in the school system, as it does in organisations like hospitals or other work environments. However, the teaching load of a teacher never covers a full day; therefore, it is possible to think of a day's schedule as being split in "flexible" shifts. The first shift always starts right at the first time period of a day and the last one always finishes with the last time period. Intermediate shifts may also exist but this depends on the number of teachers and their weekly teaching load. The length of the daily shifts varies for the different teachers of a school and it is formed according to the teaching needs. An additional advantage of the shift assignment stage is that teachers that need to teach simultaneously, collaboratively or in parallel are assigned from the beginning to the same shift and this makes the final schedules more satisfactory.

For the actual timetabling problem, at the second stage, we concentrate in modelling all the requirements that result from school regulations and some well-accepted quality rules as hard constraints. The rest of the requirements, which refer mainly to the precedence of core courses in the daily schedules, the preferences of teachers and the compactness of their schedules, are considered as soft (desired) and are left for the objective function. Therefore, our two-stage approach devotes quite an effort in assigning to the cost coefficients of the objective function those values that will assure a closer proximity to the most desired solutions. The result of this process is timetables of high quality, meaning that teachers have weekly schedules that are as compact as possible, teacher preferences are satisfied as much as possible, and core courses are taught early in the day.

The paper is structured as follows. Section 2 gives a brief literature review on the timetabling problem in general and the school timetabling problem in particular. Emphasis is given to the Integer Programming approaches. Section 3 gives the details of our modelling approach, the definition of the work shift for the educational environment and the integer programming models developed for the two stages. The case study of a real school is provided in section 4 to demonstrate the usability of our approach. Further experimentation with five different schools of varying size is also attempted in order to study the influence of various factors to the quality of the resulting timetables. Finally, in section 5, we summarise our approach and provide useful conclusions drawn from it.

2.0 Background and motivation

> "*Timetabling* represents the most important planning exercise in the school calendar. It not only gives practical expression to the *curricular philosophy* of the school, it sets, maintains and regulates the *teaching* and *learning pulse* of the school and ensures the *delivery of quality education* for all students" [Learning and Teaching Scotland, 2006].

This statement, written by educators, emphasizes the importance of the timetabling process for the educational system and highlights the multiple objectives of the task. Moreover, it makes clear that while the timetabling practice requires adopting all requirements and constraints that hold uniquely for each institution, quality is an ill-defined feature that every institution strives to achieve.

A theoretical version of the school timetabling problem has been formulated as a class/teacher mathematical model (de Werra, 1985). In that model, pairs of classes and teachers (called lectures) are assigned to time periods to satisfy a requirement matrix and in such a way that no two teachers or two classes are involved with the same class or teacher, respectively, during the same time period. In such form, the problem is always solvable as long as the required teaching load for any teacher or class does not exceed the length of the timetable (Even et al., 1976). Moreover, in this version or in a couple of its simple variants, the problem is reduced to an edge colouring graph problem and can be solved in polynomial time. The addition of an objective function that measures the total cost of assignments to an originally search problem, provides one way to measure quality (Junginger, 1986). As it is argued below, quality is a multi-attribute feature and the total cost of assignments in the objective function is usually a weighted average of the important quality attributes.

The school timetabling problem has been formulated in many and very different ways and has been solved using several analytical or heuristic approaches: Graph colouring (Neufeld and Tartar, 1974; Cangalovic and Schreuder, 1991), network flow techniques (Ostermann and de Werra, 1983), genetic algorithms

(Colorni et al., 1990; 1998; Drexl and Salewski, 1997), constraint programming (Valouxis and Housos, 2003), simulated annealing (Abramson, 1991), column generation (Papoutsis, et al., 2003), tiling algorithms (Kingston, 2005) and other local search algorithms (Costa, 1994; Schaerf, 1999b). Of course, the timetabling requirements for the secondary educational level are very diverse for the different countries; therefore, every model has limited use.

In (Lawrie, 1969) an IP model has been developed to formulate the timetabling problem for high schools in UK. The model follows a structure developed by the same author uniquely for the UK educational system, where the curriculum of pupils is given in terms of layouts. The IP model in this case is solved using the Gomory's method of integer forms, while the objective function does not take into account any of the desired quality features. The lack of efficient software tools for the solution of IP models a few years back has forced researchers away from suggesting standard IP solution techniques. It is more than a decade, however, that this obstacle has been removed and mathematical programming tools are contributing towards the effort of automating the timetabling process for high schools (Birbas et al., 1997a) and for the universities (Tripathy, 1992; Dimopoulou and Miliotis, 2001; Daskalaki et al., 2004; Daskalaki and Birbas, 2005; Avella and Vasilev, 2005; Schimmelpfeng and Helber, 2007).

While the school timetabling problem is known to be NP-complete (de Werra, 1997), enhancing a model with real-world practical or quality features increases even more the complexity of the problem. For example, compactness is a type of constraint that is usually required in school timetabling for the students but not necessarily for the teachers. The former is specifically modelled as a hard constraint in several models presented in the literature (Drexl and Salewski, 1997; Birbas et al., 1997a, 1997b, and 1999; Schaerf, 1999b; Valouxis and Housos, 2003; Papoutsis et al., 2003). Theoretically, deciding whether a timetable that guarantees compactness for the students exists is NP-complete by itself (Asratian and de Werra, 2002). However, in most practical situations, the total number of time periods scheduled for each class is much larger than the total number of time periods scheduled for each teacher; therefore, it is almost certain that there are plenty of feasible solutions that satisfy the student compactness requirement. In contrast, compactness for teachers is not usually required but in several cases it is strongly preferred and recognized as a quality feature (Valouxis and Housos, 2003; Papoutsis et al., 2003). In fact, the two compactness requirements may be conflicting to each other. Therefore, it is important to search for compact timetables for teachers only after compactness for students has been fully satisfied. While it is easier to achieve both in situations like the typical "class-teacher problem", things become worse when additional realistic requirements are added. Additional sources of complexity include special scheduling requirements like the simultaneous or collaborative teaching and parallel courses. In such cases, it is almost impossible to find fully compact schedules for teachers and the quality of the resulting timetables degrades (Asratian and de Werra, 2002; de Werra et al., 2002).

An additional source of complexity in timetabling problems results from requirements for consecutive time periods. Such a constraint, which is very realistic, by itself, turns timetabling problems from polynomially solvable to NP-hard (ten Eikelder and Willemen, 2001). Especially in university timetabling, where a great percentage of the courses require blocks of lectures with two, three or even more consecutive time periods, it is quite beneficial to relax this constraint first and try to patch the solution afterwards in some way (Daskalaki and Birbas, 2005). This approach improves tremendously the required CPU time and it is therefore recommended for university environments. However, in school timetabling the consecutive time periods represent just a small percentage of the courses, thus causing no major problem. For the school timetabling problem it is essential to achieve compact schedules for the teachers along with other quality features, thus our approach gives primary importance to this issue.

Beyond school timetabling, the literature in the general area of timetabling and rostering appears quite rich (Burke and Rudova, 2007; Burke and Trick, 2005; Burke and Gausmaecker, 2003; Burke and Erben, 2001; Burke and Carter, 1998; Burke and Ross, 1996). The University or Course Timetabling Problem (Carter and Laporte, 1998; Burke and Petrovic, 2002; Petrovic and Burke, 2004) and the Exam Timetabling Problem (Carter and Laporte, 1996; Carter et al., 1996) represent the most closely related subjects and share similar solution approaches; however, they carry significant differences and therefore are treated separately by the researchers. Apart from mathematical programming, mentioned earlier, representative solution approaches to these problems also include graph colouring and its variants (Burke et al., 2003b), tabu search and hyper-heuristics (Hertz, 1992; Burke et al., 2003a), constraint-based techniques (Deris et al., 1997) and case-based reasoning (Burke et al., 2003c; Burke et al., 2006).

3.0 Modelling approach and problem formulation

The following basic assumptions are set forth to describe the framework for our models.

1) The school works for I days every week (Monday through Friday) and for J time periods every day of the week.
2) The school comprises a number of classes, which break into K class-sections, in total, $\{A_1, A_2 \ldots B_1, B_2 \ldots C1, C_2 \ldots\}$.
3) There are L teachers available for teaching. Most of them work full time, but there is always a percentage of part timers. The actual teaching load of a teacher depends on his/her seniority and the needs for courses within his/her specialty.
4) Every class-section k requires a timetable that includes all courses designed for it.
5) All students of a given class-section attend the same courses during most of the time periods. For a small number of time periods, however, some class-sections split into smaller sections or reshuffle for attending certain "special" courses.
6) Most courses require sessions of at most one time period per day, however, certain courses may require blocks of more than one consecutive time periods.
7) Classrooms are assumed pre-assigned to class-sections. With the exception of certain lab work and a small number of "special" courses, the students attend courses in their dedicated classroom.

Assumption (5) refers to courses that require special care for their scheduling. In this model we handle the following cases:

- *Simultaneous teaching of different courses from two teachers to two different groups of students*. Depending on the school, offered courses like *Computers* or *Physics* or *Technology* may require lab work. If the lab room cannot fit all students of a section, then two sub-sections are formed. One sub-section attends the underlined course (m_1), taught by teacher l^*, while the other sub-section attends another course (m_2) with similar requirements (Fig. 1a). According to our modelling approach, for every such pair of teachers one is considered to be the *basic* and the other the *non-basic*. For this specific assignment the non-basic teacher follows the schedule of the basic.

- *Collaborative teaching of the same course by two teachers and for the same group of students*. Again for courses that require lab work, if the lab room can fit all students of the section, it is not uncommon to assign more than one teacher to be present in the room (Fig. 1b). Like in simultaneous teaching, for every pair of teachers that are assigned concurrently to the same section, one is considered to be the *basic* and the other the *non-basic.*
- *Parallelism of courses*. Courses on foreign languages often are designed for at least two levels (beginners / advanced). Assuming that there is only one English teacher, then students from two different class-sections are joined together to form a new section just for one time period and attend the beginners level course. The rest of the students from both class-sections form another section and attend a different course, for example *Physical Education* (Fig. 1c). Afterwards, all students return back to their original class-section to continue with their regular schedule. The difference of this scenario from the previous ones is the reshuffling of two class-sections. Still, one of the teachers will be named as the *basic* one and the other the *non-basic.*

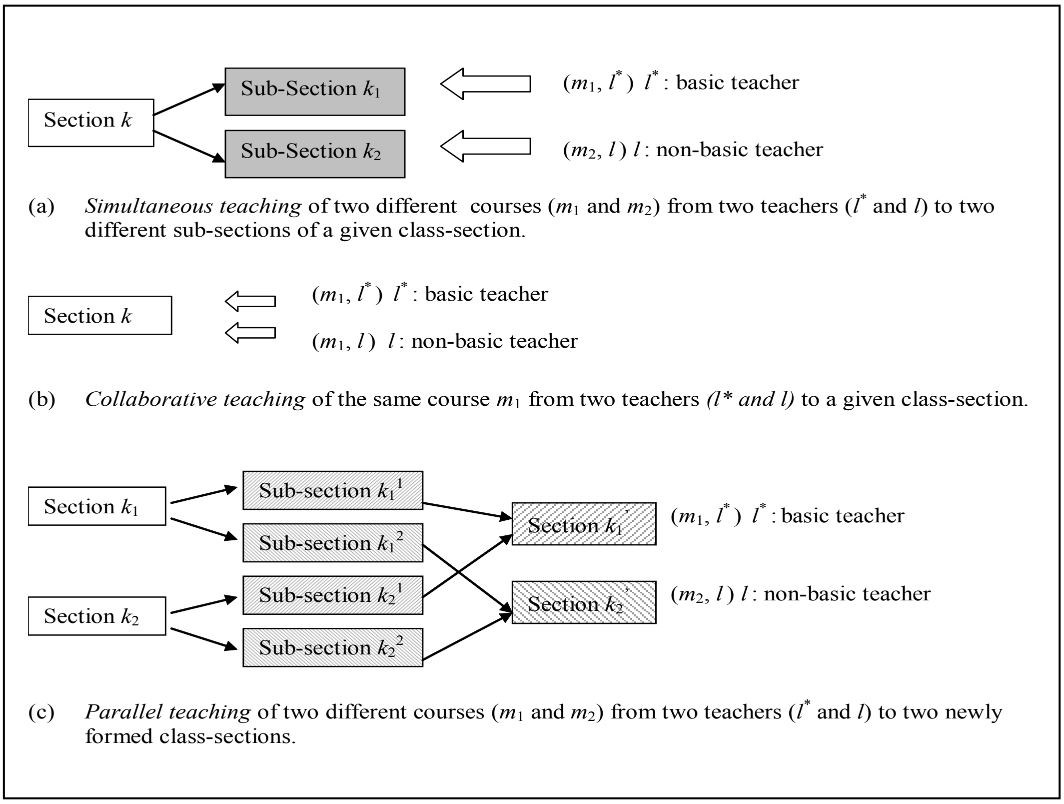

Figure 1: Courses that require special treatment in scheduling

A look in the timetabling literature reveals that in many school systems there are requirements for parallel courses and for simultaneous or collaborative teaching (de Gans, 1981; Schaerf, 1999b); therefore, such features are very important for the timetabling models, even though they do add significant complication to the models.

As explained earlier our proposed solution consists of two stages. In the first stage, the Shift Assignment Problem (SAP) is solved, using an IP model. The SAP takes as input the teachers′ preferences for a specific part of each day (declared as early, middle or late shift), the requirements of the school for simultaneous or collaborative teaching or parallel courses and the school policy for core courses. In the second stage, the actual Timetabling Problem (TP) is solved using a different IP model, while the values of the cost coefficients in the objective function are determined with the help of the output from the previous stage.

The objective of the SAP is the assignment of teachers to a minimal number of work shifts during each day of the week, so that the weekly schedule of the whole school is feasible. It takes into account the fact that teachers never have a full-day teaching assignment and that they prefer compact, instead of sparse, daily schedules. The assignment of teachers to work shifts obeys the following rules:

- Every teacher is assigned to exactly one work shift for each day of presence in the school.
- Depending on his/her specialty, each teacher teaches either mostly core courses or mostly non-core courses. The first group of teachers should be assigned preferably to early work-shifts, while the second group should rather be assigned to later ones. If a teacher teaches both core and non-core courses, then this affects the order of the course assignments within a given work shift.
- When it is required from the school, two or sometimes even three specific teachers may have to be assigned to the same work shift, so simultaneous, collaborative or parallel teaching can be possible.
- During any time period and as a result during any work shift there should be at least so many teachers present as there are class-sections in the school.

On the contrary, the TP takes care of all scheduling rules and regulations of the educational system, which are modelled as constraints of the IP model. More specifically, the model provides timetables that carry the following characteristics:

- They assign at most one course to each teacher for every time period.
- They are compact for the students, i.e. there are no empty slots in the student schedules.

- They are complete in the sense that they cover all courses required by the student curriculum and for the required amount of time periods per course.
- The timetable of each class-section is balanced, in the sense of “time spent at school” throughout the week.
- All full time teachers are assigned teaching assignments for each day of the week, in order to guarantee daily presence. Part time teachers, respectively, are assigned teaching load only for those days that are available to the particular school.
- All courses are scheduled for at most one time period per day, with the exception of those courses that specifically require consecutive time periods.
- Courses that require consecutive time periods are scheduled for at most one multi-period session per day and fully cover their weekly requirements.
- Occasionally certain courses require to be scheduled simultaneously and more than one teacher should be scheduled for the same course.

Apart from the hard constraints that the models handle, the following soft quality requirements are also considered:

- Core courses should be scheduled early in the day
- Teacher preferences should be satisfied as much as possible
- Teacher schedules should be as compact as possible

In order to create a timetable for a school, the two models are solved sequentially. Starting from the SAP and the optimal work shifts assignments for each teacher during the week, it is then easier to find an optimal solution for the TP that distributes the courses taught by the given teacher to time periods within the shifts that the specific teacher is assigned to be present. For the solution of the IP models in both stages the ILOG CPLEX MIP Solver has been used.

Table 1. Definition of parameters

Acronym	Name	Description
TTP	Total Time Periods	*The total number of time periods to be scheduled in the timetable*
WTL_l	Weekly Teaching Load for teacher l	*The total number of time periods to be assigned to teacher l each week*
$\overline{DTL_l}$	Average Daily Teaching Load for teacher l	*The daily average number of time periods assigned to teacher l*
WTL_{kl}	Weekly Teaching Load of teacher l for class-section k	*The total number of time periods to be assigned to teacher l for class-section k each week*
WTL_{klm}	Weekly Teaching Load of teacher l for class-section k and course m	*The total number of time periods to be assigned to teacher l for course m each week*
TTP_k	Total Time Periods for section k	*The total number of time periods over all courses of class-section k to be assigned each week*

Prior to any model presentation, Table 1 depicts the acronyms, names and descriptions of certain parameters that are used further down.
It is clear that the following feasibility conditions should hold:

$$(1)\ TTP = \sum_{l=1}^{L} WTL_l = \sum_{k=1}^{K} TTP_k$$

$$(2)\ WTL_l = \sum_{k=1}^{K} WTL_{kl} = \sum_{k=1}^{K}\sum_{m=1}^{M} WTL_{klm}$$

$$(3)\ TTP_k = \sum_{l=1}^{L} WTL_{kl} = \sum_{l=1}^{L}\sum_{m=1}^{M} WTL_{klm}$$

$$(4)\ WTL_{kl} = \sum_{m=1}^{M} WTL_{klm}$$

The feasibility conditions (1) – (4) hold exactly as they appear above only for those schools that carry no special requirements for the scheduling of their courses. When requests for simultaneous, collaborative or parallel scheduling are present, one should consider only the basic teachers for each such assignment.

3.1 Definition of work shifts for teachers

The notion of work shifts in this problem is somewhat different than the shifts as being used in other work environments, for example hospitals or supermarkets. The main disparity is the flexibility that should be maintained for the starting and/or ending time of each work shift that is assigned to an individual. Let us assume for a while that there are no courses with special scheduling requirements. Then the following proposition must hold for every feasible solution:

Proposition 1. *At any time period of a given day, there must be at least as many teachers present in the school as the number of class-sections*.

The proposition must be true; otherwise, at least one class-section would not be assigned a course during the specified time period, a situation that violates the requirement for compact student schedules.

In order to generate work shifts for the teachers of a school, the concept of the "equivalent-of-a-teacher" (EOT) is introduced. During any day of the week the teaching load of a teacher is always less and sometimes much less than J, the number of time periods of a day, because

$$1 \le \overline{DTL_l} < J, \quad \forall l \in \mathcal{L}.$$

As a result, if we allow the courses that the given teacher teaches to be scheduled on any time period of the day, the introduction of empty slots in his/her timetable is unavoidable. The number of empty slots in the teachers' schedules can be minimal if we consider an "ideal" operation of the school, as follows: On the first period of the day, there are exactly so many teachers present as the number of class-sections. The teachers proceed with their individual teaching schedules, with no idle period in-between, until one or more of them finish for the day. For every teacher that completes his/her teaching activities, another one starts until someone else takes over or until the day finishes. The changeover between two teachers may occur after any period of the day. The subset of teachers that work in a sequential manner for a day will be called the "*equivalent-of-a-teacher*" (EOT). We can further think of the notion of the "*virtual class-section*" for the school, so that an EOT is assigned to one and only one virtual class-section for the whole day. Then a similar proposition, like the one before, holds for the EOTs and the virtual class-sections:

Proposition 2. *On a given day in a school there should be exactly as many EOTs as the number of virtual class-sections.*

This proposition must also be true for the compactness criterion to be satisfied. In this sense, the timetabling turns into an assignment problem.

Tables 2, 3 and 4 demonstrate further the aforementioned idea. Table 2 presents a possible one-day schedule with seven time periods for a school that consists of four sections and eight teachers. Teachers 1 and 4 form an EOT by sharing the teaching load for virtual section #1, i.e. the virtual section #1 is assigned to the subset of teachers {1, 4}. Likewise, the subsets of teachers {3, 8}, {6, 2} and {7, 5} form three more EOTs, which are assigned to virtual sections #2, #3, and #4, respectively. In reality, each teacher switches from one real class-section to another, so the right part of Table 2 illustrates a possible assignment of the eight teachers to the actual class-sections of the school (let's say A1, A2, B1, and C1). Since the sections are four and the teachers eight (a multiple of four), for every EOT the working day is split in two shifts of different lengths. The work shifts for a school cannot be rigid in nature, but flexible enough to absorb peculiarities of the educational system.

Table 2. A typical day schedule for a school with 8 teachers and 4 sections.

<table>
<tr><th></th><th>P1</th><th>P2</th><th>P3</th><th>P4</th><th>P5</th><th>P6</th><th>P7</th><th></th><th>P1</th><th>P2</th><th>P3</th><th>P4</th><th>P5</th><th>P6</th><th>P7</th></tr>
<tr><td>Teach1</td><td colspan="4">v_section 1</td><td colspan="3"></td><td></td><td>A1</td><td>B1</td><td>C1</td><td>A2</td><td></td><td></td><td></td></tr>
<tr><td>Teach2</td><td colspan="3"></td><td colspan="4">v_section 3</td><td></td><td></td><td></td><td></td><td>C1</td><td>A2</td><td>A1</td><td>B1</td></tr>
<tr><td>Teach3</td><td colspan="2">v_section 2</td><td colspan="5"></td><td></td><td>B1</td><td>C1</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Teach4</td><td colspan="4"></td><td colspan="3">v_section 1</td><td></td><td></td><td></td><td></td><td></td><td>B1</td><td>C1</td><td>A2</td></tr>
<tr><td>Teach5</td><td colspan="4"></td><td colspan="3">v_section 4</td><td></td><td></td><td></td><td></td><td></td><td>C1</td><td>A2</td><td>A1</td></tr>
<tr><td>Teach6</td><td colspan="3">v_section 3</td><td colspan="4"></td><td></td><td>C1</td><td>A2</td><td>B1</td><td></td><td></td><td></td><td></td></tr>
<tr><td>Teach7</td><td colspan="4">v_section 4</td><td colspan="3"></td><td></td><td>A2</td><td>A1</td><td>A1</td><td>B1</td><td></td><td></td><td></td></tr>
<tr><td>Teach8</td><td colspan="2"></td><td colspan="5">v_section 2</td><td></td><td></td><td></td><td>A2</td><td>A1</td><td>A1</td><td>B1</td><td>C1</td></tr>
</table>

Table 3. A typical day schedule for a school with 9 teachers and 4 sections.

<table>
<tr><th></th><th>P1</th><th>P2</th><th>P3</th><th>P4</th><th>P5</th><th>P6</th><th>P7</th><th></th><th>P1</th><th>P2</th><th>P3</th><th>P4</th><th>P5</th><th>P6</th><th>P7</th></tr>
<tr><td>Teach1</td><td colspan="4">V_section 1</td><td colspan="3"></td><td></td><td>A1</td><td>B1</td><td>C1</td><td>A2</td><td></td><td></td><td></td></tr>
<tr><td>Teach2</td><td colspan="3"></td><td colspan="4">v_section 3</td><td></td><td></td><td></td><td></td><td>C1</td><td>A2</td><td>A1</td><td>B1</td></tr>
<tr><td>Teach3</td><td colspan="2">v_section 2</td><td colspan="5"></td><td></td><td>B1</td><td>C1</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Teach4</td><td colspan="4"></td><td colspan="3">v_section 1</td><td></td><td></td><td></td><td></td><td></td><td>B1</td><td>C1</td><td>A2</td></tr>
<tr><td>Teach5</td><td colspan="4"></td><td colspan="3">v_section 4</td><td></td><td></td><td></td><td></td><td></td><td>C1</td><td>A2</td><td>A1</td></tr>
<tr><td>Teach6</td><td colspan="3">v_section 3</td><td colspan="4"></td><td></td><td>C1</td><td>A2</td><td>B1</td><td></td><td></td><td></td><td></td></tr>
<tr><td>Teach7</td><td colspan="4">V_section 4</td><td colspan="3"></td><td></td><td>A2</td><td>A1</td><td>A1</td><td>B1</td><td></td><td></td><td></td></tr>
<tr><td>Teach8</td><td colspan="2"></td><td colspan="3">v_section 2</td><td colspan="2"></td><td></td><td></td><td></td><td>A2</td><td>A1</td><td>A1</td><td></td><td></td></tr>
<tr><td>Teach9</td><td colspan="5"></td><td colspan="2">v_section 2</td><td></td><td></td><td></td><td></td><td></td><td></td><td>B1</td><td>C1</td></tr>
</table>

Table 4. A typical day schedule for a school with 7 teachers and 4 sections.

<table>
<tr><th></th><th>P1</th><th>P2</th><th>P3</th><th>P4</th><th>P5</th><th>P6</th><th>P7</th><th></th><th>P1</th><th>P2</th><th>P3</th><th>P4</th><th>P5</th><th>P6</th><th>P7</th></tr>
<tr><td>Teach1</td><td colspan="5">v_section 1</td><td colspan="2"></td><td></td><td>A1</td><td>B1</td><td>C1</td><td>A2</td><td>A1</td><td></td><td></td></tr>
<tr><td>Teach2</td><td colspan="3"></td><td colspan="4">v_section 2</td><td></td><td></td><td></td><td></td><td>C1</td><td>A2</td><td>A1</td><td>B1</td></tr>
<tr><td>Teach3</td><td colspan="3">v_section 2</td><td colspan="4"></td><td></td><td>B1</td><td>C1</td><td>A1</td><td></td><td></td><td></td><td></td></tr>
<tr><td>Teach4</td><td colspan="4"></td><td colspan="3">v_section 3</td><td></td><td></td><td></td><td></td><td></td><td>B1</td><td>C1</td><td>A2</td></tr>
<tr><td>Teach5</td><td colspan="3"></td><td colspan="4">v_section 4</td><td></td><td></td><td></td><td></td><td>B1</td><td>C1</td><td>A2</td><td>A1</td></tr>
<tr><td>Teach6</td><td colspan="4">v_section 3</td><td colspan="3"></td><td></td><td>C1</td><td>A2</td><td>B1</td><td>A1</td><td></td><td></td><td></td></tr>
<tr><td>Teach7</td><td colspan="3">v_section 4</td><td></td><td></td><td colspan="2">v_sect 1</td><td></td><td>A2</td><td>A1</td><td>A2</td><td></td><td></td><td>B1</td><td>C1</td></tr>
</table>

Similarly, Table 3 exposes an assignment of shifts for a school with four class-sections and nine teachers. In this example, the subsets of the teachers {1, 4}, {3, 8, 9}, {6, 2} and {7, 5} form the EOTs. It is noted that since $[9/4]=2$ and $9 \bmod 4 = 1$, three EOTs are composed of two teachers and exactly one of three. Therefore, the maximum number for the work shifts is three, however only one teacher is assigned to the third shift. For the examples of Tables 2 and 3, if course requirements for the teachers and classes permit, the teachers may not have idle periods in their schedule.

The last example (Table 4) refers to a school with seven teachers and four class-sections. This time $[7/4]=1$ and $7 \bmod 4 = 3$; therefore, one teacher must carry a full-day assignment and the others will form pairs to share the day. However, if there are not enough courses for a single teacher to fill a work shift of seven time periods his/her schedule inherently holds idle time periods. According to this example the four EOTs are {1, 7}, {3, 2}, {6, 4} and {7, 5}, i.e. teacher #7 participates in two EOTs and in some respect it looks as if he/she is assigned two shifts.

The number and length of work shifts in any given school depends on the number of teachers and their availability for teaching (part-time/full-time), the number of class-sections and the number of courses. The notion of work-shifts lies in the heart of the model to be presented in the next section. As suggested by the three examples, for fully compact teacher schedules every teacher should be assigned to only one shift. However, additional constraints and planning of the actual courses required by the educational system often violates this desirable feature. Our approach aims at keeping the number of violations at a minimum level.

3.2 Modelling the Shift Assignment Problem (SAP)

Two sets of binary variables, one with basic and the other with auxiliary variables are used for modelling the SAP. Variable $x_{i,l,b}$ takes the value of 1, when teacher $l \in \mathcal{L}$ is scheduled to work during shift $b \in \mathcal{B}$ of day $i \in \mathcal{I}$. Auxiliary variables are introduced only when there is a need for forcing one teacher to be in the same shift with another teacher to serve in parallel sessions. Therefore, the variable $y_{i,l_1,l_2,b}$ takes the value of 1 when teachers $l_1 \in \mathcal{L}$ and $l_2 \in \mathcal{L}$ are both assigned to shift $b \in \mathcal{B}$ of day $i \in \mathcal{I}$.

3.2.1 Constraints for the SAP

Five sets of constraints are set forth for this model.

α) Uniqueness

Every teacher of the school should be assigned to exactly one shift for each day:

$$\sum_{b\in\mathcal{B}} x_{i,l,b} = 1, \qquad \forall\ i\in\mathcal{I},\ \forall\ l\in\mathcal{L}_i \tag{3.1}$$

where $\mathcal{L}_i = \{\ l\in\mathcal{L}:\ l$ is a teacher available to the school on day $i\in\mathcal{I}\}$.

The constraints ensure daily presence of every teacher for all days for which he/she is scheduled to be in that school.

b) School policy for the shift assignments

The weekly assignments of a given teacher to a specific shift (e.g. morning shift, middle shift, etc.) cannot exceed a pre-defined upper limit:

$$\sum_{i\in\mathcal{I}} x_{i,l,b} \le r_{lb}, \qquad \forall\ l\in\mathcal{L},\ \forall\ b\in\mathcal{B} \tag{3.2}$$

where r_{lb} is the maximum number of shifts b allowed per week for teacher l. This limit reflects the policy of each school and may depend on the teacher's specialty.

c) Pre-assignment of shifts

Under certain circumstances a teacher may be allowed the pre-assignment of a shift:

$$x_{i,l,b} = 1, \qquad \forall\ (i,\ l,\ b)\in \mathcal{B}_{fix} \tag{3.3}$$

where $\mathcal{B}_{fix} = \{(i,\ l,\ b)$: teacher $l\in\mathcal{L}$ shall be assigned to shift $b\in\mathcal{B}$ of day $i\in\mathcal{I}\}$.

d) Conjugation of shifts

(i) Teachers l_1 and l_2 shall be assigned to the same shifts because their teaching assignments comprise courses that shall be taught in parallel. This rule is necessary when there is a need for simultaneous teaching, collaborative teaching or parallel teaching. The constraints needed in this case are:

$$x_{i,l_1,b} - x_{i,l_2,b} \le 0, \qquad \forall i\in\mathcal{I},\ \forall b\in\mathcal{B},\ \forall (l_1,l_2)\in V_1 \cup V_2 \ \wedge\ q_{l_1} \le q_{l_2} \tag{3.4a}$$

where

$V_1 = \{(l_1,l_2)$: l_1 and l_2 are teachers that need to teach simultaneously to different sub-sections or collaboratively to the same section$\}$,

$V_2 = \{(l_1,l_2)$: l_1 and l_2 are teachers teaching in parallel to two different class sections$\}$

and q_{l_i} indicate the weekly time-period requirement for the course that the teacher l_i (i = 1, 2) teaches and requires conjugation of shifts.

Constraints (3.4a) safeguard the assignment of teachers l_1 and l_2 to the same shift on a given day.

$$x_{i,l_1,b} - x_{i,l_2,b} - y_{i,l_1,l_2,b} \leq 0, \qquad \forall i \in \mathcal{I},\ \forall b \in \mathcal{B},\ \forall (l_1, l_2) \in V_1 \wedge q_{l_1} \leq q_{l_2} \qquad (3.4b)$$

Constraints (3.4b) are implemented only in the case of simultaneous or collaborative teaching for linking the auxiliary variables $y_{i,l_1,l_2,b}$ with $x_{i,l,b}$.

$$\sum_{i \in \mathcal{I}} \sum_{b \in \mathcal{B}} y_{i,l_1,l_2,b} = q_{l_1} \qquad \forall (l_1, l_2) \in V_1 \wedge q_{l_1} \leq q_{l_2} \qquad (3.4c)$$

Lastly, constraints (3.4c) ensure that exactly q_{l_1} shifts are allocated to teacher l_1, who is the teacher of the course with the smallest requirement in terms of time periods per week.

(ii) Teachers l_1 and l_2 shall be assigned to the same shifts because their teaching assignments comprise courses that shall be taught in parallel; however, teacher l_3 also requires simultaneous teaching with l_1 during some other periods. In this case, the teachers l_2 and l_3 are assigned to consecutive shifts:

$$x_{i,l_1,b} - \sum_{b=b-1 \vee b=b+1} x_{i,l_3,b} \leq 0, \quad \forall i \in \mathcal{I},\ \forall b \in \mathcal{B},\ \forall\ (l_1; l_2, l_3) \in V_3 \qquad (3.4d)$$

where

$V_3 = \{(l_1; l_2, l_3)$: l_1 is required to teach some course at the same time with teacher l_2 and another course at the same time with teacher $l_3\}$

Constraints (3.4d) allow the simultaneous teaching of teachers l_1 and l_2 for certain time periods and afterwards the simultaneous teaching of teachers l_1 and l_3.

e) Completeness

During any shift of the day, the number of teachers required to be present shall be equal to the number of class-sections of the school. For the cases of collaborative teaching for a given course, this number changes, necessitating the following constraints:

For all work-shifts of any given day, except of the last one, the following constraints are set forth:

$$\sum_{l \in \mathcal{L}} x_{i,l,b} - \sum_{(l_1,l_2) \in V_1} y_{i,l_1,l_2,b} = K, \qquad \forall\ i \in \mathcal{I},\ \text{and}\ b = 1, \ldots, B-1 \qquad (3.5a)$$

where K is the number of the class-sections and B is the number of the daily shifts.

For the last shift B, however, only the remaining teachers may be assigned:

$$\sum_{l\in\mathcal{L}} x_{i,l,B} + \sum_{(l_1,l_2)\in V_l} \sum_{b\in\mathcal{B}-\{B\}} y_{i,l_1,l_2,b} = \upsilon, \qquad \forall\ i\in\mathcal{I} \tag{3.5b}$$

where υ is defined as $L \bmod K$ and the number of shifts is defined as the ceiling of L/K, i.e. $B = \left\lceil \frac{L}{K} \right\rceil$.

3.2.2 Objective function

The objective function for this problem is a linear cost function:

$$\text{Minimise } \sum_{i\in I}\sum_{l\in L}\sum_{b\in B} c_{i,l,b}x_{i,l,b} \tag{3.6}$$

The coefficients $c_{i,l,b}$ are defined in a way that reflects the teachers' preferences for specific shifts for all days. The lower the value for $c_{i,l,b}$, the higher the preference for shift *b*. An example of such a value system appears in Table 5. A high value for a particular cost coefficient prevents, but does not prohibit the assignment of its corresponding variable to the final solution.

Table 5. Possible values for the cost coefficients $c_{i,l,b}$

Rank of preference	Cost of Preference
First preference	50
Second preference	120
Third preference	200

3.3 Modelling the Timetabling Problem

The timetabling problem, as it is modelled in this section, takes responsibility for the assignment of subjects, teachers and class sections to the time periods of a week, taking into account all functional rules imposed by a given school system, typical for the Hellenic educational system and many other countries. Furthermore, the solution of the SAP guides the process of assigning values to the cost coefficients in the objective function based on the shifts that each teacher is assigned for each day of the week.

Two sets of binary variables are again employed. The first consists of the basic variables denoted by $x_{i,j,k,l,m}$, where $i\in\mathcal{I}, j\in\mathcal{J}$, $k\in\mathcal{K}, l\in\mathcal{L}$, and $m\in\mathcal{M}$. $x_{i,j,k,l,m}$ is assigned the value of 1, when course *m*, taught by teacher *l* to the class-section *k,* is scheduled for the *j*th period of day *i.* The second set consists of the auxiliary variables $y_{i,t_m,k,h_m,m}$, used only for those courses, which require sessions of consecutive periods. $y_{i,t_m,k,h_m,m}$ takes the value of 1, when course *m,* is scheduled for h_m consecutive periods on day i for class-section k, with t_m being the 1st period for this assignment. The IP model embodies as constraints all standard scheduling rules for the timetabling process and many non-standard ones imposed from regulations of the educational system we have studied.

3.3.1. Constraints

The constraints are presented in three different groups. The first group are the *standard constraints*, named as such because they take care of rules that play a fundamental role in all timetabling problems and appear in nearly all formulations. The second group consists of several *non-standard constraints* in the sense that many of these constraints appear in timetabling problems, however not all schools operate under these rules. The third group refers to the *special scheduling requirements* that certain courses may require. Requirements like "consecutive teaching periods" result from the nature of certain courses, while others result from an effort for the educational system to be more flexible and adaptable to students' needs.

Throughout the presentation of the constraints, a number of sets – usually subsets of the main sets – are utilized in order to limit down the number of equations and the number of variables in the model. These sets are defined as follows:

$\mathcal{M}_{kl}$ ={ $m \in \mathcal{M}$: m is a course that teacher l teaches for class-section k}

$\mathcal{M}^*_{kl}$ ={ $m \in \mathcal{M}$: m is any regular course that teacher l teaches for class-section k}, where the term "regular" refers to all courses that do not require any special type of scheduling.

$\mathcal{M}^{sim}_{l}$ = {(m, k, l^*): $m \in \mathcal{M}$ is a course that teacher $l \in \mathcal{L}$ teaches for a part of section $k \in \mathcal{K}$ simultaneously with another course taught by the "basic" teacher l^*}

$\mathcal{M}^{col}_{l}$ = {(m, k, l^*): $m \in \mathcal{M}$ is a course that teacher $l \in \mathcal{L}$ teaches for section $k \in \mathcal{K}$ in collaboration with the "basic" teacher l^*}

$\mathcal{M}_{cons}$ = {(m, k, h_m): $m \in \mathcal{M}$ is a course of section $k \in \mathcal{K}$ that needs to be scheduled in block(s) of h_m consecutive periods}

$\mathcal{M}_{paral}$ = {[(m_a, m_b); (k_a, k_b); (l_a, l_b)]: $m_a \in \mathcal{M}$ is a course that teacher l_a teaches for section k_a that needs to be scheduled always in parallel to m_b a course taught by teacher l_b for section k_b }

$\mathcal{M}_{excl}$ = {[(m_a, k_a, l_a; m_b, k_b, l_b; ...;m_w, k_w, l_w) (l_a)]: $m_a, m_b, ..., m_w \in \mathcal{M}$ are courses that teachers $l_a, l_b, ..., l_w \in \mathcal{L}$ teach to sections $k_a, k_b, ..., k_w \in \mathcal{K}$, respectively, however no more than a certain number of them may be scheduled in the same day or time period}

$\mathcal{M}_{fix}$ = {(i_a, j_a, m_a, k_a, l_a): $m_a \in \mathcal{M}$ is a course that teacher l_a teaches to section k_a and should be scheduled on day i_a and in period j_a}

$\mathcal{I}_l$ = { $i \in \mathcal{I}$: i is any day of the week for which teacher l is available for the school}

$\mathcal{K}_l$ = { $k \in \mathcal{K}$: k is a class-section of the school to which teacher l teaches at least one course}

In addition, we define the maximum stretch, a parameter that is used in the model:

Definition: *Maximum Stretch* for class-section k, denoted by $S_k^{\max}$, is the maximum number of teaching periods that section k may have during any day of the week.

$S_k^{\max}$ may be set equal to J, the length of each day, however, in order to create more balanced timetables for the classes, it is preferable to set a different upper limit for each section of the school. Therefore, $S_k^{\max}$ equals to $\left\lceil \frac{TTP_k}{I} \right\rceil$. In general, however, it holds that $S_k^{\max} \le J, \quad \forall k \in \mathcal{K}$.

A. Standard Constraints

The constraints of this group make sure that the resulting timetables are conflict-free and contain all courses required for scheduling.

A1. Uniqueness

Every teacher may be assigned at most one course and one class-section in a given period with the exception of indicated courses that require more than one instructor.

$$\sum_{k \in \mathcal{K}_l} \sum_{m \in \mathcal{M}_{kl}^*} x_{i,j,k,l,m} + \sum_{(m,k,l^*) \in \mathcal{M}_l^{sim} \cup \mathcal{M}_l^{col}} x_{i,j,k,l^*,m} \le 1, \quad \forall l \in \mathcal{L}, \forall j \in \mathcal{J}, \forall i \in \mathcal{I}_l \tag{3.7}$$

A2. Completeness for students

All courses in the curriculum of a class-section should appear in the timetable for the required number of teaching periods.

$$\sum_{l\in\mathcal{L}_k}\sum_{m\in\mathcal{M}_{kl}}\sum_{i\in\mathcal{I}_l}\sum_{j\in J}x_{i,j,k,l,m}+\sum_{l\in\mathcal{L}_k}\sum_{(m,k,l^*)\in\mathcal{M}_l^{sim}\cup\mathcal{M}_l^{col}}\sum_{i\in I_l}\sum_{j\in J}x_{i,j,k,l^*,m}=TTP_k,\quad \forall k\in\mathcal{K} \tag{3.8}$$

A3. Completeness for teachers

All courses assigned to a given teacher should appear in the timetable for the required number of teaching periods.

$$\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\sum_{m\in\mathcal{M}_{kl}^*}x_{i,j,k,l,m}+\sum_{(m,k,l^*)\in\mathcal{M}_l^{sim}\cup\mathcal{M}_l^{col}}\sum_{i\in\mathcal{I}_l}\sum_{j\in\mathcal{J}}x_{i,j,k,l^*,m}=WTL_{kl},\quad \forall k\in\mathcal{K},\forall l\in\mathcal{L} \tag{3.9}$$

A4. Completeness for courses

The teaching periods assigned to a given course over a whole week should add up to the weekly requirements for the specific course.

$$\sum_{i\in\mathcal{I}_l}\sum_{j\in\mathcal{J}}x_{i,j,k,l,m}=WTL_{klm}\ \ \forall k\in\mathcal{K},\ \forall\ l\in\mathcal{L}_k,\forall m\in\mathcal{M}_{kl}, \tag{3.10}$$

In a given model, not all of the completeness constraints are needed. The feasibility conditions in the beginning of section 3.0 may guarantee that only the last (Eq. 3.10) is sufficient to satisfy all others.

B. Non-Standard (Operational) Constraints.

In this group of constraints several rules that quite often are requested from the educational systems are presented.

B1. Compactness for students

The timetable of every class-section should not carry empty slots during the week

$$\sum_{l\in\mathcal{L}_k}\sum_{m\in\mathcal{M}_{kl}^*}x_{i,j,k,l,m}+\sum_{(m,k,l^*)\in\mathcal{M}_l^{sim}\cup\mathcal{M}_l^{col}}x_{i,j,k,l^*,m}=1,\quad \forall k\in\mathcal{K},\ i\in\mathcal{I},\ j=1,\ldots,(S_k^{\max}-1) \tag{3.11a}$$

$$\sum_{l\in\mathcal{L}_k}\sum_{m\in\mathcal{M}_{kl}^*}x_{i,S_k^{\max},k,l,m}+\sum_{(m,k,l^*)\in\mathcal{M}_l^{sim}\cup\mathcal{M}_l^{col}}x_{i,S_k^{\max},k,l^*,m}\le 1,\ \ \forall\ i\in\mathcal{I},\ \forall k\in\mathcal{K} \tag{3.11b}$$

$$\sum_{l\in\mathcal{L}_k}\sum_{m\in\mathcal{M}_{kl}^*}\sum_{j\in\mathcal{J}}x_{i,j,k,l,m}+\sum_{(m,k,l^*)\in\mathcal{M}_l^{sim}\cup\mathcal{M}_l^{col}}\sum_{j\in\mathcal{J}}x_{i,j,k,l^*,m}\le S_k^{\max},\ \forall i\in\mathcal{I},\forall k\in\mathcal{K} \tag{3.12}$$

Constraints (3.11) ensure that for each section there is exactly one course scheduled for any given period (except may be the last one) of any day. Constraints (3.12), on the other hand, checks whether all courses of section k are scheduled within the maximum stretch allowed for the section. The requirement for compact student schedules appears in many school timetabling problems and this is a major difference between university and school timetabling practice.

B2. Daily presence of teachers

Many schools require daily presence from the full-time teaching staff in the school, with the exception of the part-time staff that declare ahead of time the days of availability. Therefore, every teacher should be assigned at least one teaching period of each day that is available to the specific school.

$$\sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}_l}\sum_{m\in\mathcal{M}^*_{kl}} x_{i,j,k,l,m} + \sum_{(m,k,l^*)\in\mathcal{M}^{sim}_l\cup\mathcal{M}^{col}_l}\sum_{j\in\mathcal{J}} x_{i,j,k,l^*,m} \geq 1, \quad \forall\, l\in\mathcal{L},\ \forall\, i\in\mathcal{I}_l, \qquad (3.13)$$

Not all schools carry such a requirement. Instead, some schools allow for one day-off in the week or they carry some different policy for presence in the school.

B3. Uniform distribution of courses

Any given course may be scheduled for at most one teaching period per day of the week, unless it requires more teaching periods than the days of the week or there is a special request for multiple or consecutive hours, in which cases they are scheduled accordingly.

$$\sum_{j\in\mathcal{J}} x_{i,j,k,l,m} \leq 1, \qquad \forall\, i\in\mathcal{I},\ \forall\, k\in\mathcal{K},\ \forall\, l\in\mathcal{L}_k,\ \forall\, m\in\mathcal{M}_{kl} \qquad (3.14)$$

C. Special Requirements

This last group of constraints are provided to handle special requirements that sometimes appear in connection with certain courses. These requirements increase the complexity of the problem. In our case, it does increase significantly the number of equations and the number of variables. However, we have applied this IP modelling for many schools in our country of quite varying sizes and still the problem was solvable.

C1. Consecutiveness of teaching periods

The school timetabling process should accommodate special requirements for certain courses to be taught in multi-period slots at most once a day for a given class section.

Let m be a course that requires a session of h_m consecutive periods and l the teacher that teaches the course for section k. Then the following should apply:

$$h_m * y_{i,t_m,k,h_m,m} \leq \sum_{j=t_m}^{t_m+h_m-1} x_{i,j,k,l,m} \leq y_{i,t_m,k,h_m,m} + h_m - 1, \quad \forall (m,k,h_m)\in M_{con}, \forall i\in\mathcal{I}_l,\ t_m \leq J - h_m + 1$$

(3.15a)

$$\sum_{t_m=1}^{J-h_m+1} y_{i,t_m,k,h_m,m} \leq 1 , \qquad \forall\ (m, k, h_m) \in \mathcal{M}_{con},\ i \in \mathcal{I}_l \qquad (3.15b)$$

$$\sum_{i \in I_l} \sum_{t_m=1}^{J-h_m+1} y_{i,t_m,k,h_m,m} = v_m , \qquad \forall\ (m, k, h_m) \in \mathcal{M}_{con} \qquad (3.16)$$

where v_m is the number of the multi-period slots required for course m every week.

Constraints 3.15a force h_m basic variables $x_{i,j,k,l,m}$ that refer to consecutive time periods to take the value of 1, while constraints 3.15b ensure that only one block of consecutive periods may be assigned in any given day. Finally, constraint 3.16 indicates that there should be exactly v_m of these blocks for the whole week.

C2. Parallelism of courses

As explained in the Introduction, this constraint is activated when the students of two different sections (k_1 and k_2) are re-arranged in order to form two new sections that attend two different courses (m_1 and m_2) assigned to teachers l_1 and l_2, respectively. So, it is only needed to schedule the two courses m_1 and m_2 for the same time periods.

$$x_{i,j,k_1,l_1,m_1} - x_{i,j,k_2,l_2,m_2} \leq 0, \quad \forall\ [(m_1, m_2); (k_1, k_2); (l_1, l_2)] \in \mathcal{M}_{par},\ \forall\ i \in \mathcal{I},\ \forall j \in \mathcal{J} \qquad (3.17)$$

C3. Mutual exclusiveness of courses

Sometimes it is desirable for two or more courses to be scheduled at different time periods or days or more generally only up to n_r of them to be scheduled at the same time. In this case the following constraints are appropriate.

$$\sum_{(m,k,l) \in \mathcal{M}_{excl}} x_{i,j,k,l,m} \leq n_r , \qquad \forall\ i \in \mathcal{I}\ \forall j \in \mathcal{J} \qquad (3.18)$$

Given there is no index for the classrooms in the decision variables, these constraints are important in order to avoid clashes for the laboratories or specially equipped rooms.

C4. **Pre-assignment of certain courses**

Lastly, pre-assignments of certain courses to specific time periods are easily handled by the following constraints.

$$x_{i,j,k,l,m} = 1, \qquad \forall\ (i, j, k, l, m) \in \mathcal{M}_{fix} \qquad (3.19)$$

3.3.2 Objective function

The objective function for the timetabling problem is a cost function, with two distinct terms. The cost coefficients $c_{i,j,k,l,m}$ represent the cost of assigning course m

taught by teacher l for section k to period j of day i. Similarly, the cost coefficients $a_{i,t_m,k,h_m,m}$ refer to the cost of scheduling course m for section k on day i and for the periods t_m up to t_m+h_m-1, given that course m requires h_m consecutive time periods. Therefore, the objective of the IP model is to minimize the function:

$$\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\sum_{l\in\mathcal{L}}\sum_{k\in\mathcal{K}_l}\sum_{m\in\mathcal{M}_{kl}} c_{i,j,k,l,m}x_{i,j,k,l,m}+\sum_{i\in\mathcal{I}}\sum_{(m,k,h_m)\in\mathcal{M}_{cons}}\sum_{t_m\in\mathcal{P}_{t_m}} a_{i,t_m,k,h_m,m}y_{i,t_m,k,h_m,m} \tag{3.20}$$

where P_{t_m} = { $t_m\in\mathcal{J}$: t_m is a possible value for the 1st time period of the h_m consecutive time periods for course $m \in \mathcal{M}_{cons}$}

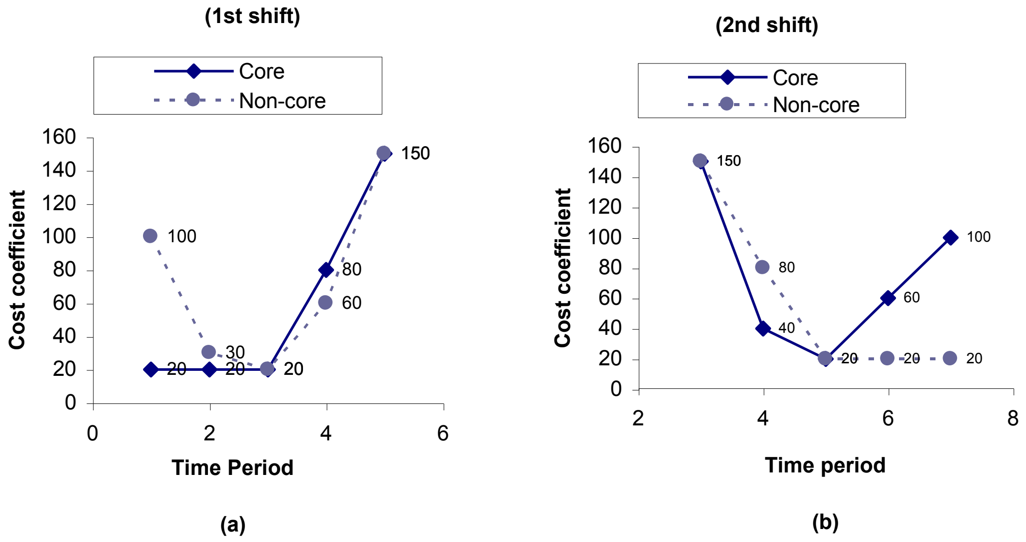

Figure 2. Penalty functions for assigning values to cost coefficients in a 2-shift problem

The cost coefficients $c_{i,j,k,l,m}$ take values with the help of the output from the SAP and certain penalty functions like the ones shown in Figs. 2 and 3. Fig. 2 shows an example of such penalty functions for a school that operates with two shifts per day. For a given day and for a core course, if the teacher is assigned to the first shift (by the SAP) then the corresponding cost coefficient (in the TP) will be assigned the values indicated by the solid line in Fig. 2a for the different time-periods of the day. Since it is preferable for the core courses to be scheduled as early as possible during the day, we provide low values for the first three periods, medium values for the fourth period and high or very high (not shown in the figure) for the rest of the day. If the solution from the SAP assigns the teacher to the second shift, then the corresponding cost coefficients may take the values of the solid line in Fig. 2b. For the second shift, the fifth and fourth periods are preferable for the core courses and are assigned the lowest values, while the sixth and seventh are in the medium-low range. The rest of the periods are assigned very high values (not shown) in order to avert assignments of core courses to these periods. On the contrary, for a non-core course, if the teacher is assigned to the first or the second shift then the corresponding cost coefficients may be assigned the values indicated by the dotted lines in Figs. 2a and 2b, respectively.

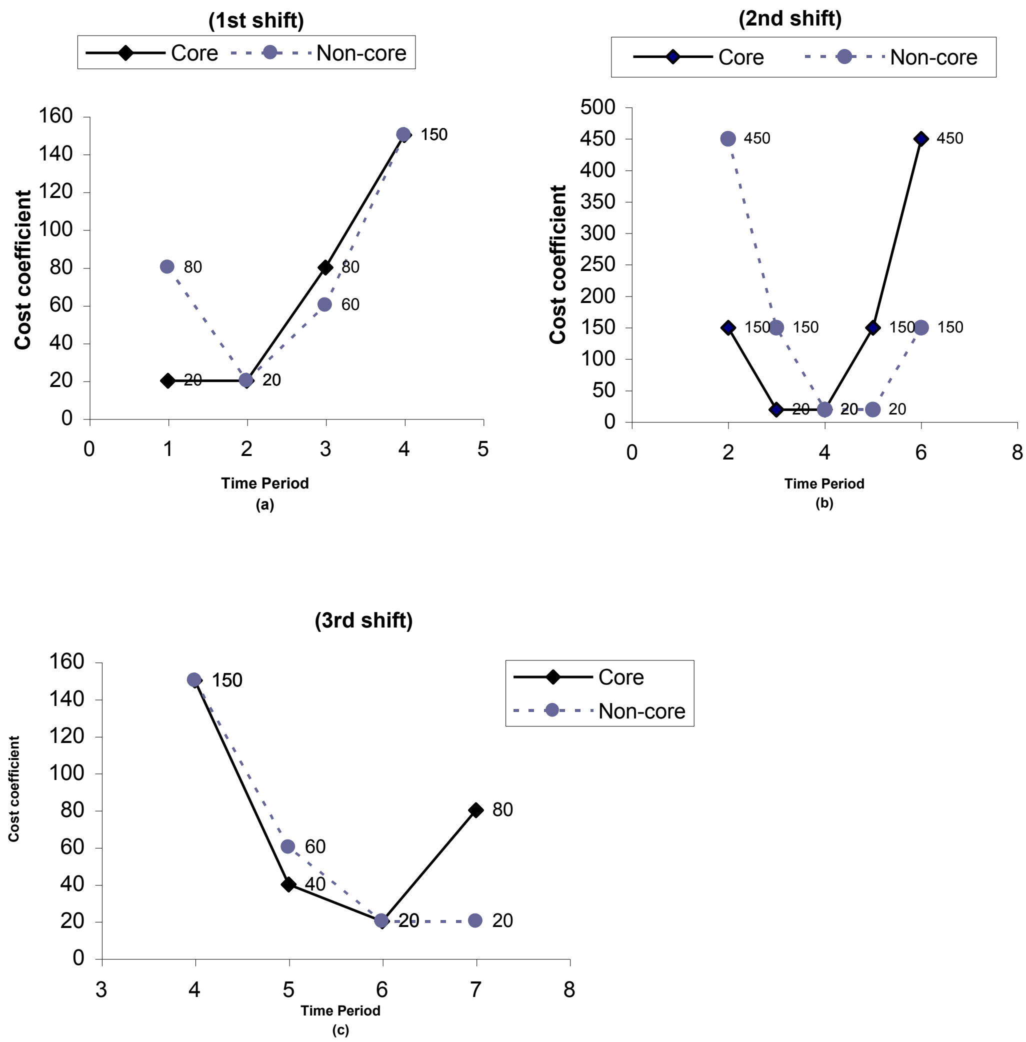

Figure 3. Penalty functions for assigning values to cost coefficients in a 3-shift problem

Similarly, Fig. 3 depicts a similar set of penalty functions that could be used for assigning values to the cost coefficients of the TP objective function when a school operates with three shifts for the teaching staff. The rationale behind these functions is similar to those in Fig. 2. Finally, Fig. 4 displays an example penalty function that could be used for assigning values to the $a_{i,t_m,k,h_m,m}$ cost coefficients of the TP objective function for a course that requires two consecutive time periods for its teaching. The chart suggests that it is most preferable for this course to start either on the second time period, otherwise on the third or fifth period. It is quite clear that all penalty functions may take different shapes and each school may introduce its own policy for the desired assignments.

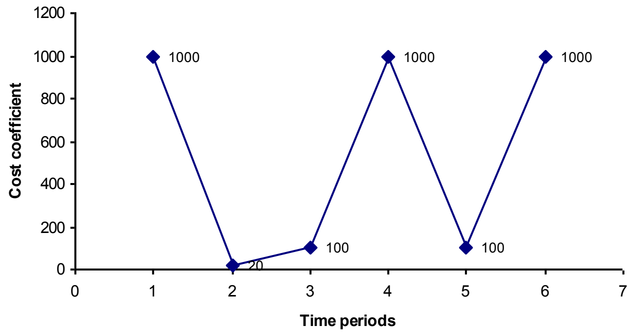

Figure 4. Penalty function for assigning values to cost coefficients for courses that require two consecutive time periods

4.0 Experimental Results

In order to demonstrate the effectiveness of the modelling approach just presented, the results of a case study are shown in this section. The study concerns an existing secondary school with six class-sections, two sections for each class, i.e. {A1, A2, B1, B2, C1, and C2}. Twenty-three teachers are serving in the school, sixteen full-time and seven part-time. For the timetable of this school, 95 courses ought to be assigned to 5 days of the week, with 7 periods per day. The problem has been solved using the approach presented in section 3 and the results are presented below. The solution from the SAP indicated the work shifts, where each teacher should be assigned for each day of the week, based on individual preferences, the needs of the school and the policy about core and non-core courses. Then, using the penalty functions of Fig. 3, the values for the coefficients in the objective function of the TP were assigned and the problem was solved. For both problems, the MIP solver of ILOG CPLEX 10.1 has been used.

Table 6. The teachers' timetable for the case study problem.

DAYS	MONDAY							TUESDAY							WEDNESDAY						
TEACHERS	1^{η}	2^{η}	3^{η}	4^{η}	5^{η}	6^{η}	7^{η}	1^{η}	2^{η}	3^{η}	4^{η}	5^{η}	6^{η}	7^{η}	1^{η}	2^{η}	3^{η}	4^{η}	5^{η}	6^{η}	7^{η}
T1	B2	A1	A2										C1	C2					B2	A2	
T2	C1	C1								C1	C2								C1	C1	C2
T3						A1				B2	A2	B2					B2	B2			
T4			A1	C2	C1							C2	A1	C1				A1			
T5	A1	C2						C2	C1						C2	A1					
T6					A2		B1	A2	B2	A2							A2	A2	B1		
T7			B1	B1								B1	C2		B1	B1					
T8	A2	B1	C1					C1	A2						C1	A2	B1				
T9					A1	C2	B2		C2	A1	B2						A1	C2			
T10	***	***	***	***	***	***	***						A2	A2	***	***	***	***	***	***	***
T11	C2	B2								B1	C1				B2	C1					
T12			B2	C1						C2	B1								C2	A1	C1
T13 (French)				A1	B2	A2		B1	A1						A2	B2					
T14 (French)						C1	C2	***	***	***	***	****	***	***			C2	C1			
T15 (German)					B2	A2	C2	***	***	***	***	****	***	**	A2	B2	C2				
T16 (Computers)	B1	A2	C2					B2	B1						A1	C2	C1				
T17 (Technology)	B1	A2						B2	B1						A1						
T18 (Arts+Crafts)			C2	B2	C2						A1	C1				C2	C1	B1			
T19				A2				A1													B1
T20 (English)						B1	A1					A2	B2	B1					A2	B2	A1
T21 (Phys.Educ)					B1	B2	A2					A1	B1	B2					A1	B1	A2
T22 (Phys.Educ)	***	***	***	***	***	***	***	***	***	***	***	****	***	***	****	***	***	***	***	***	***
T23							C1							A1						C2	B2

DAYS	THURSDAY							FRIDAY						
TEACHERS	1^{η}	2^{η}	3^{η}	4^{η}	5^{η}	6^{η}	7^{η}	1^{η}	2^{η}	3^{η}	4^{η}	5^{η}	6^{η}	7^{η}
T1	B1	A1										B1	C2	C1
T2				C1						C1	C1			
T3	B2		B2	A2				B2	A1		B2			
T4			C2	A1	C1			A1		C2				
T5					A1	C2					A1	C1		
T6	A2	A2							A2	B2	A2			
T7				B1	C2					B1	B1	C2		
T8		B1	C1					A2	B1					
T9	A1	B2									C2	B2		
T10	****	***	***	***	***	***	***						A2	A2
T11			B1	C2	B2			B1	B2					
T12					B1	A1	A1					A1	C1	C2
T13 (French)					A2	B2	B1						A1	B1
T14 (French)						C1	C2	***	***	***	***	***	***	***
T15 (German)					A2	B2	C2	***	***	***	***	***	***	***
T16 (Computers)							C1					A2	B2	A1
T17 (Technology)	****	***	***	***	***	***	***					A2	B2	A1
T18 (Arts+Crafts)						A2	C1	***	***	***	***	***	***	***
T19							B2						B1	B2
T20 (English)	C1	C2	A2					C2	C1	A1				
T21 (Phys.Educ)			A1	B2						A2				
T22 (Phys.Educ)	C2	C1						C1	C2					
T23						B1	A2	***	***	***	***	***	***	***

According to Table 6, which depicts the teachers' view of the timetable, the daily load of each teacher is quite compact. For example, teacher T1 is assigned to teach for three successive periods on the first shift of Monday. Similarly, on Tuesday teacher T1 teaches during the third shift for two successive periods. Since compactness for teachers is only a soft constraint, idle periods cannot be avoided completely; so teachers T6 and T4 carry one idle period each and teacher T3 carries two idle periods, marked with black boxes in the table. The stars on some teachers' schedules, on the other hand, indicate that these teachers are working part-time for the school and are not available for these specific days.

Moreover, in Table 6, one may observe the scheduling of courses with special requirements. For example, teacher T13, who is teaching *French* and teacher T15, who is teaching *German*, are both assigned section B2 on the 5th period on Monday. In this way, the two teachers have been scheduled to teach simultaneously *French* and *German* to two different sub-sections of section B2. It is noted that the students of Hellenic gymnasiums are required to study a third language, which can be *French* or *German*, besides the mandatory *Modern Greek* and *English*; therefore, none of the students can be registered in both sub-sections. In addition, teachers T16 and T17, who teach the courses of *Computers* and *Technology*, respectively, teach simultaneously their courses by splitting one section (e.g. B1) into two sub-sections. During a given session, half of the class is scheduled for *Computers* and the other half for *Technology*, while during another session the two sub-sections interchange subjects. Moreover, teacher T16 teaches a subsection of C2 simultaneously with teacher T18 who teaches *Arts and Crafts* to the other subsection of C2. Thus, T16 and T17 are scheduled for the first shift and T18 for the second, so that all simultaneously taught courses take effect.

Lastly, the notion of parallel teaching is applied to teachers T20 and T21, who teach *English* and *Physical Education*, respectively. On Monday, during the 6th period, sections B1 and B2 are assigned to those teachers. Students from both sections, who are in the "beginners" class of *English* form a new section and attend the class of teacher T20, while at the same time, the rest of the students have their *Physical Education* course. Similarly, on Tuesdays, again during the 6th period these two sections are again scheduled to meet these two teachers. This time, however, the students who are in the "advanced" class of *English* attend the course of teacher T20 and the rest of the students the *Physical Education* course. Moreover, class B is required to have three time periods of *Physical Education* per week and only two of *English*; therefore, teacher T21 has to meet each one of the two sections (B1 and B2) for one time period in addition to those of teacher T20. This requirement is also covered effectively in the output of the TP model.

We conclude commenting on the results of this case study with the presentation of the EOTs that are formed for each day of the week (Table 7). Since there are six class-sections, exactly six EOTs are formed. The available teachers vary from 19 to 21 for the different days of the week; therefore, most EOTs are formed with three teachers and only a few with four. From each pair of teachers that teach simultaneously, only one participates in the formation of an EOT and the other one is indicated next to its respective *basic teacher* separated with a slash (/). For example, on Monday the teachers 1, 13, and 23 form an EOT and split the day into three shifts, while teacher 15 is also scheduled for the second shift to teach simultaneously with teacher 13. It is worth noting that while teachers 13, 7 and 19, for example, are all assigned to the second shift on Monday, the actual time periods that they teach do not necessarily coincide. From Table 6, the second shift for T13 points to periods {4, 5 and 6}, while for T7 to {3, 4} and for T19 to {4}. Lastly, the teachers that are forced to carry idle periods in their schedules appear in more than one EOT. For example, teacher 6 carries one idle time period on Monday and therefore appears in both EOT #5 and EOT #6. In Table 7 these teachers are indicated with underlined and bold values.

Table 7. The EOTs for each day of the week as formed by the TP model

A/A	Monday	Tuesday	Wednesday	Thursday	Friday
1	{1, 13/15, 23}	{5, 2, 4}	{5, 3, 1, 19}	{1, 4, 23}	{**3**, **4**, 5, 13}
2	{2, 4, 20}	{6, 18, 1}	{7, 6, 23}	{**3**, 8, 2, 12}	{8, 7, 10}
3	{5, 7, 9}	{8, 3, 10}	{8, 4, 12}	{6, 11, 14/15}	{11, 2, 1}
4	{8, 19, 21}	{13, 11, 7, 23}	{11, 9, 12}	{9, **3**, 5, 19}	{20, **3**, 12}
5	{11, 12, **6**, 14/15}	{16/17, 12, 20}	{13/15, 14/15, 20}	{20, 7, 18/16}	{22, 21, 9, 19}
6	{17/16, 18/16, 3, **6**,}	{19, 9, 21}	{17/16, 18/16, 21}	{22, 21, 13/15}	{**4**, 6, 16/17}

Furthermore, the shift assignments that are indicated in Table 7 through the EOTs are not always the same with the shifts that the SAP had assigned to the teachers. This is the case because the TP model solves the problem as a whole; therefore, it is only guided but not constrained by the solution of the SAP model. Finally, from this table it is easy to detect the teachers that carry empty slots in their schedules. These teachers appear in two different EOTs in a given day, for example, T6 on Monday, T3 on Thursday, etc. Effectively these teachers are assigned two shifts on those days and this is again an intervention of the TP model that violates the "one-shift only" requirement in the SAP due to course scheduling requirements.

4.1 Quantification of quality in timetables

According to the results from the case study, our modelling approach has managed to effectively handle all hard and soft requirements set forth to us. However, measuring the degree of satisfaction for each specific soft constraint is a question that has not been answered yet. For this reason, our study proceeded further, towards the quantification of quality for the resulting timetables. The requirements that have been set forth to measure quality are the following:

Q1: *Core courses, if possible, should be scheduled during the early periods of each day, when students are more alert and their concentration more manageable.*
Q2: *The preferences of the teachers for early, intermediate or late shift should be satisfied as much as possible.*
Q3: *Given that the daily teaching requirements for the teachers do not cover the whole day, it is preferred that teachers' schedules are as compact as possible, i.e. the total number of empty slots in teachers' schedules should be minimal.*

These three quality requirements have been characterised as important to the school timetabling problem and have influenced indirectly the optimisation process. In order to further measure the satisfaction level, three quality indices are proposed:

1. *Core-course Scheduling* S*atisfaction* (*CSS*), to measure the percentage of core courses scheduled during early time periods (1^{st} to 4^{th}).

2. *Teachers Preference Satisfaction* (*TPS*), to measure the percentage of assignments that match the teachers' pre-defined preferences.

3. *Teachers Schedule Compactness* (*TSC*), to measure the percentage of the teaching periods assigned to teachers over the total number of time periods of presence effectively needed from the teachers (i.e. the sum of teaching periods and idle periods).

As we have already discussed in section 3.0, quality requirements Q1 and Q2 conflict with each other, if the corresponding teachers' preferences for specific shifts do not go along with Q1. For this reason, constraint 3.2 in the SAP model has been added to set an upper limit to the satisfaction of teachers' preferences, if this is necessary. To test the performance of our timetabling approach according to the quality requirements, an additional extensive study has been carried out involving five different schools of varying sizes. Table 8 shows the size of the schools used for this study. The smallest (School #1) has 5 sections, 13 teachers (two of which are part-time) and a total of 170 time periods to be assigned to courses. Respectively, the largest (School #5) has 21 sections, 48 teachers (seven of which are part-time) and 721 time periods for scheduling. For Hellenic standards this is pretty much the range of school size in the country. For all schools, the timetabling approach presented in section 3.0 was applied and the corresponding SAP and TP models were built. The sizes of the TP models are presented in Table 8 using {# of variables} and {# of constraints}.

Table 8. The size of schools and models used for experimentation

School	# of sections	# of teachers	# of time periods	# of variables	# of constraints
1	5	13 (2 p-t)	170	2135	1190
2	6	16 (5 p-t)	202	2380	1287
3	9	21 (6 p-t)	306	3885	1958
4	12	29 (6 p-t)	404	5089	2595
5	21	48 (7 p-t)	721	10038	4754

The quality of the resulting timetables depends on several factors; however it is definitely affected from courses with special scheduling requirements, from the school's decision whether to apply a certain policy for core and non-core courses, as well as from the actual "balance" in the teachers' preferences. In order to study the influence of the aforementioned factors along with the effectiveness of the proposed modelling approach for solving the timetabling problem, twelve different scenarios were run for each school. More specifically, we tested the following *four* combinations:

1. Courses require only simple assignments and the school has no particular policy for the core courses, so it is left on teachers' preferences whether to satisfy Q1.
2. Courses require only simple assignments and the school postulates a certain policy for core courses, so teachers' preferences are limited when necessary.
3. There are courses that require special treatment in scheduling and the school has no particular policy for the core courses.
4. There are courses that require special treatment in scheduling and the school postulates a certain policy for the core courses.

In conjunction with the above, *three* sets of preferences for the teachers were considered in order to study how the mix of teachers' preferences influences the quality of the solution.

1. Balanced preferences, i.e. half of the teachers prefer the first shift and the other half prefer the last shift.
2. All teachers prefer the first shift
3. All teachers prefer the last shift

For the five schools in our study, 60 different experiments were totally run to study the influence of the "balance" in teachers' preferences, the school policy for core courses, the special scheduling requirements, and the size of the school on the three quality indices described above, along with the ability of our approach to smooth out these factors and do the best for each case. The results from this effort are summarised in Figs. 5 - 7. Fig. 5 refers to the *TPS* index, Fig. 6 to the *TSC* and Fig. 7 to the *CSS*. For all figures, the horizontal axis refers to the school size, denoted by the number of class sections, while the vertical axes measure the corresponding index as a percentage.

Figure 5 suggests that for the simpler case (only courses with simple scheduling requirements and no school policy for the core courses) the resulting timetables may reach the maximum 100% satisfaction of teachers' preferences for the balanced case and 50% satisfaction for the unbalanced cases, when the school is large. For small schools, timetabling is easier but less flexible, thus the *TPS* index drops to approximately 85% and 45%, respectively. Requirements for special scheduling assignments for certain courses force the *TPS* index to drop significantly ranging from approximately 62% to 92% for the balanced preferences and from 35% to 50% for the unbalanced preferences. Again, we observe an upward trend of the satisfied teachers in relation to the school size, even though it may be influenced by other factors also. In addition to the special scheduling requirements, a school policy for the core courses may also reduce the achieved percentage of satisfied teachers. The size of the school turns to be a remedy in this case. However, when both courses with special scheduling requirements and school policy for the core courses co-exist, then the percentage of satisfaction is on the average around 70% and 45%, respectively for the different preference scenarios. In fact, we believe that an additional factor that plays significant role is the percentage of part-time teachers compared to those full-time. Since part-time teachers are highly constrained resources, high percentage of part-time teachers degrades quality in terms of the *TPS* index. This is the case of school #2 (with 6 classes-sections), which according to the results of Fig. 5 exhibits lower performance.

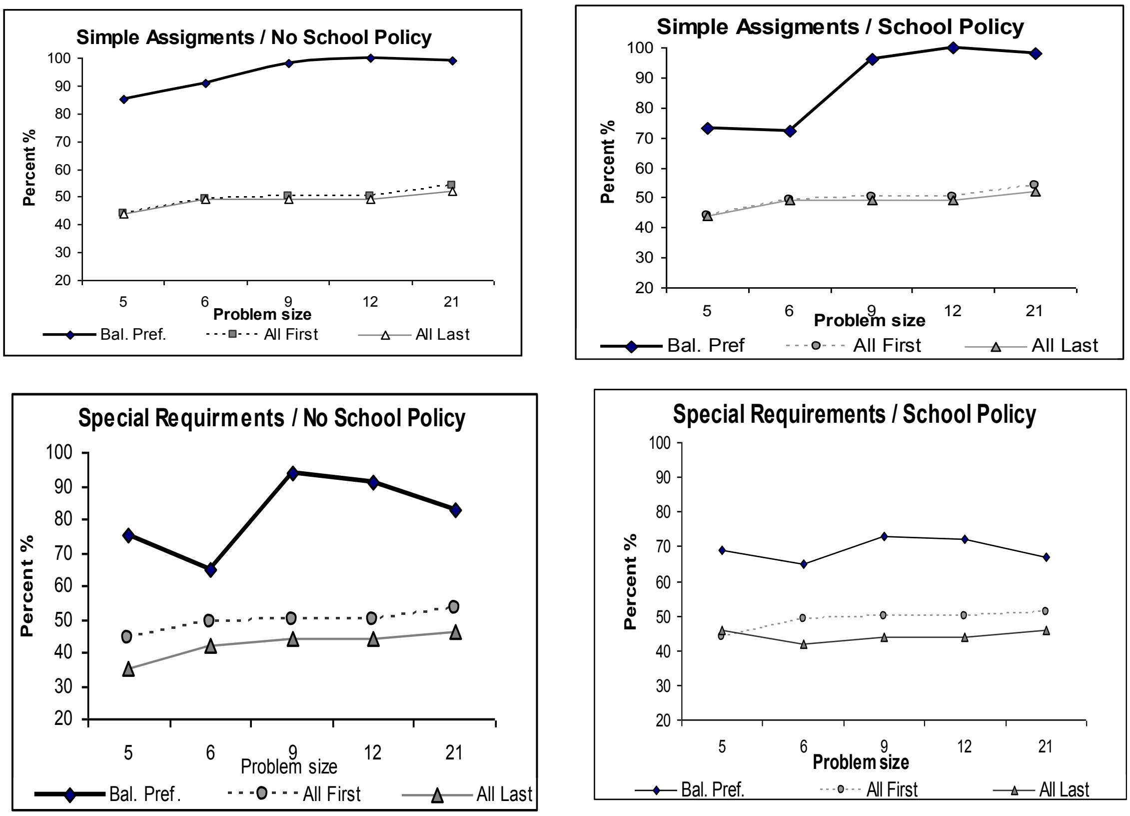

Figure 5. The Teachers' Preferences Satisfaction (*TPS*) Index.

Similarly, Fig. 6 shows the influence of the different factors we examined on the compactness of the teachers schedules. This index is almost always higher than 95% and in most cases higher than 97%. In contrast to the *TPS*, the scheme of preferences does not influence the *TSC* in some obvious way. Moreover, *TSC* in most cases degrades with the size of the school. Apparently, for the small schools (5 and 6 sections) and balanced preferences, our approach finds 100% compact timetables when only courses with simple assignments are required. Special scheduling requirements do introduce more idle periods in teacher schedules and this becomes worse as the size of the school increases.

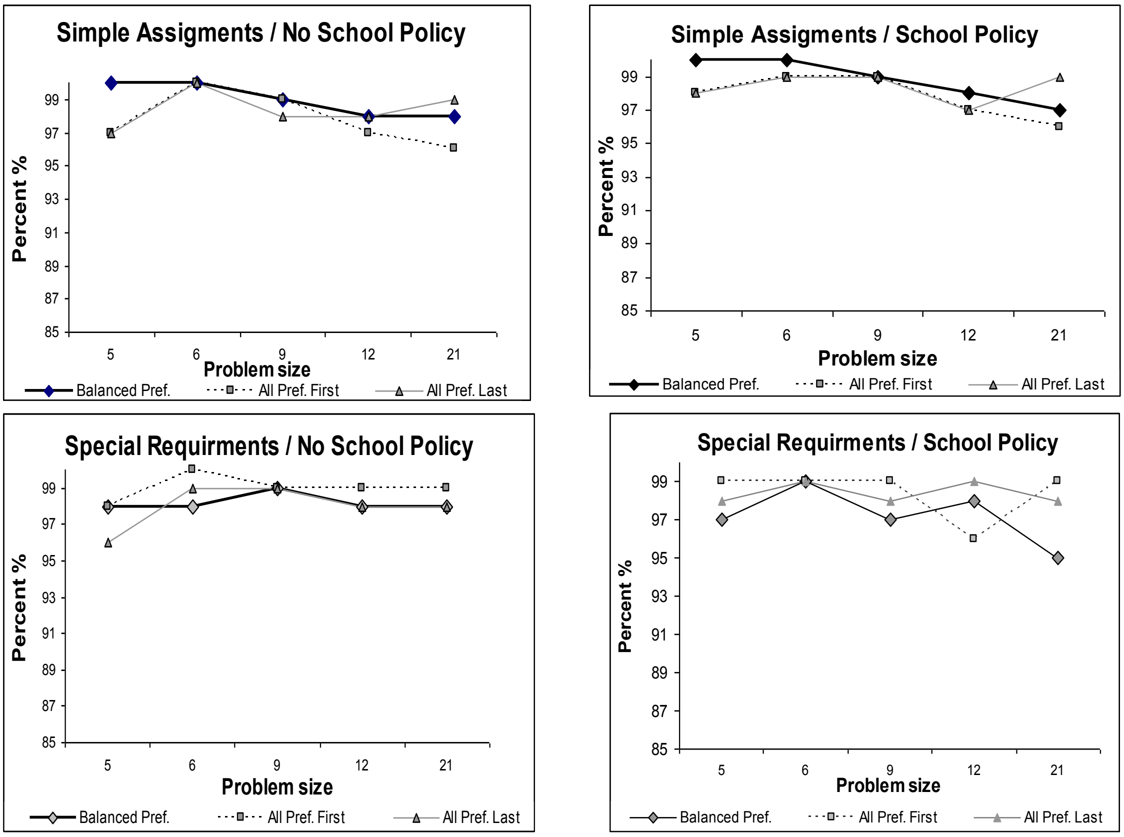

Figure 6. The Teacher Schedule Compactness (*TSC*) Index

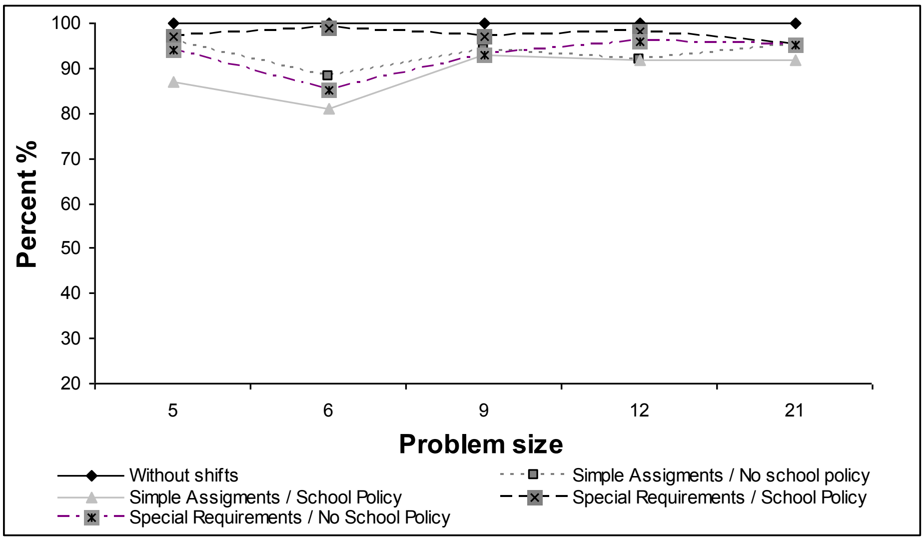

Figure 7. The Core-course Scheduling Satisfaction (*CSS*) Index

Finally, Fig. 7 shows the influence of the examined factors to the *CSS* Index. In fact, if we eliminate the Shift Assignment Problem and solve solely the Timetabling Problem taking into account only the core course precedence rule, then for the five schools we get the solid line (without shifts), which shows 100% satisfaction for all cases. *CSS* degrades as the satisfaction of teachers' preferences is introduced through the SAP, however the results show that with our approach approximately 90% of the core courses are still scheduled during the early 4 time-periods. More importantly, when special scheduling requirements are introduced for some courses and a school policy is additionally applied for the core courses, the figure shows that the *CSS* index is still very high (97% on the average).

The last thing to report from our experimentation concerns the running times of our models. Since the SAP models are small compared to the TP models and run in all cases within some fraction of a second, the CPU times that we report below refer only to the TP models. Figure 8 gives the average CPU time, measured in seconds, for each one of the five schools that we examined. All models were solved on a PC with an Intel Core 2 Duo 2.67 GHz processor, 2GB Ram and Windows XP, using the ILOG CPLEX 10.1 optimisation software. For each school the times from 12 models were averaged, while the smallest running time was 0.10 seconds and the largest 240 seconds.

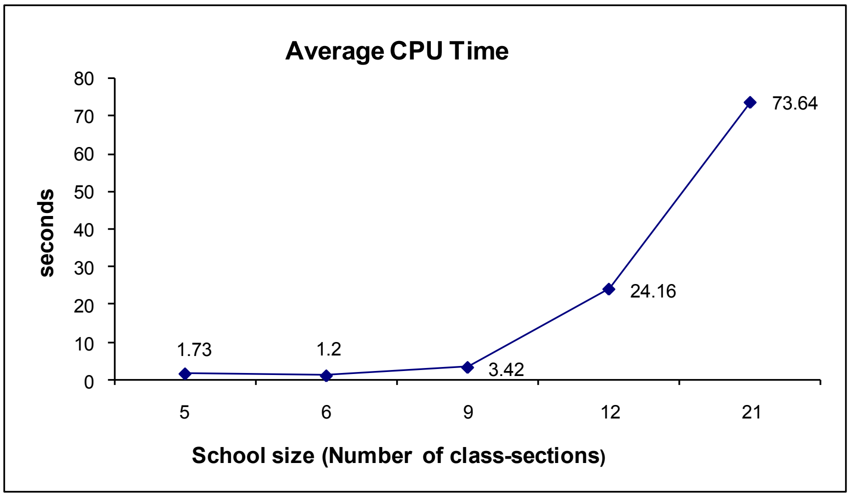

Figure 8. The average CPU time as a function of the school size

5.0 Summary and Conclusions

In this paper, we presented a novel two-stage approach for the school timetabling problem. The first stage consists of a Shift Assignment Problem, where work shifts are assigned to teachers based on their preferences, the school policy regarding core courses and the needs of the curriculum. The second stage consists of the actual Timetabling Problem resulting to an optimal timetable. In both stages Integer

Programming models are used for the solution of the corresponding problems. This approach has been applied to the Hellenic secondary educational system to achieve high-quality timetables which obey all functional and practical hard requirements and, to a very satisfactory level, a number of quality soft rules. More specifically, they are conflict free, complete, they can handle simultaneous, collaborative and parallel teaching requirements, they can schedule lectures of a given course to consecutive time-periods, they schedule courses in a balanced way across the week, and they are fully compact for the students. Still, they are also very compact for teachers; they satisfy teachers´ preferences at a high rate and at the same time force the core courses towards the beginning of each day.

In order to satisfy the compactness requirement of teachers as much as possible, the notion of *work shift* has been introduced. In addition, the notions of *virtual class-section* and *equivalent-of-a-teacher* (EOT) help visualising the teachers' schedules and indicate those subsets of teachers that share the daily teaching load for each virtual section. The shift assignments for the teachers are found with the solution of the SAP, where each teacher is assigned to one and only one shift per day. The timetabling problem is solved afterwards, and the shift assignments help us assign values for the cost coefficients in the objective function. With our approach, the values of the cost coefficients carry significant information that proves to be valuable to the timetabling procedure, which finally leads to an optimal solution that satisfies fully the hard constraints and to a high degree the soft constraints.

The approach has been applied to a large number of high schools with very satisfactory results. For experimental purposes, we chose five schools of different size. Several timetabling problems have been solved for these schools in order to study the impact of the size of the school, the mix of the preferences of the teachers, the school policy about the core courses and the requirements for simultaneous, collaborative or parallel teaching for certain courses on the quality of the resulting timetables. To make our observations measurable we establish three quality indices: the teachers´ preferences satisfaction index, an index that measures the compactness of the teachers´ schedules and finally the percentage of core courses that are scheduled during the early time periods. Our conclusions from this study are summarised as follows:

- If the courses in the timetable carry only simple requirements and there is no school policy for core courses then the indices that measure teachers´ preferences satisfaction and teachers´ schedules compactness are both very high when the teachers` preferences are balanced. In the opposite case, i.e. when teachers` preferences are totally unbalanced, the first index drops to approximately 50%, as expected, but the later remains above 95%.

- When special scheduling requirements are required from certain courses and a strict school policy is applied to limit the teachers' preferences, then we have worst-case scenarios for both indices. The teachers´ preference satisfaction index drops to approximately 70% (when the preferences are balanced) and to 45% (when the preferences are totally unbalanced). Similarly, the index that measures compactness drops slightly, but still remains almost always above 97%.
- On the contrary, the index that measures the percentage of core courses that are scheduled during the early time periods is very high (approximately 95%) when special scheduling requirements do exist and a school policy for core courses is applied.
- The size of the school influences positively the satisfaction of the teachers´ preferences and negatively their schedules compactness. However, it does not seem to affect the index for the core courses.

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